Regular local rings (All rings are Noetherian)

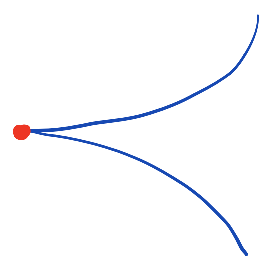
Let R be a local ring of dimension d w/ maximal ideal m.
By the PIT, m can't be generated by fewer than d elements.

m is generated by exactly d elements $\Longleftrightarrow$ it's generated by a system of parameters $x_1, \dots, x_d$

In this case, R is called regular, and $x_1, \dots, x_d$ is called a regular system of parameters.

Note that not all local rings are regular:

Ex: Let $R = \left(\mathbb{C}[x,y] / (y^2 - x^3) \right)_{(x,y)}$



(x,y) isn't principal in R,
but $(x,y)^2 = (x^2, xy, y^2) = (x^2, xy, x^3) \subseteq (x)$.

Thus, $\dim R = 1$ (since it's not 0) and x is a system of parameters.

However, if you localize at any other maximal ideal in $\mathbb{C}[x,y]/(y^2 - x^3)$ it will be regular.

In fact, a point on a scheme (or variety) is smooth $\Longleftrightarrow$ the corresponding local ring is regular!

In general, regular local rings are well behaved:

Prop: If R is a regular local ring, it's an integral domain.

Pf: let $\mathfrak{m} \subseteq R$ be the maximal ideal. We'll do induction on $\dim R$.

If $\dim R = 0$, $\mathfrak{m} = 0$, so R is a field. Now, assume $\dim R = d > 0$.

We know, by Nakayama, that $\mathfrak{m}^2 \neq \mathfrak{m}$. So by prime avoidance, we can find $x \in \mathfrak{m}$ that avoids the (finitely many) minimal primes of R and $\mathfrak{m}^2$.

Set $S = R/(x)$, and let $\mathfrak{n} = \mathfrak{m}S$ be the maximal ideal of S. Since x is [illegible] in minimal primes of R, $\dim S < \dim R$. By a previous corollary, modding out by a principal ideal in a local ring can lower the dimension by at most one, so $\dim S = d - 1$. To use induction, we need to show S is regular.

Note that $\mathfrak{n}/\mathfrak{n}^2 = \dfrac{(\mathfrak{m}/(x))}{(\mathfrak{m}/(x))^2} = \dfrac{(\mathfrak{m}/(x))}{\mathfrak{m}^2+(x)/(x)} \cong \mathfrak{m}/\mathfrak{m}^2+(x)$.

$\mathfrak{m}/\mathfrak{m}^2 \twoheadrightarrow \mathfrak{m}/\mathfrak{m}^2+(x)$ has kernel $\mathfrak{m}^2+(x)/\mathfrak{m}^2 \neq 0$.

$\mathfrak{n}/\mathfrak{n}^2$ is a $\dfrac{R/(x)}{\mathfrak{m}/(x)}$ - module. Thus it is a $R/\mathfrak{m}$ - vector space, so $\mathfrak{n}/\mathfrak{n}^2$ can be generated as an $R/\mathfrak{m}$ - module, and thus as an

S-module by $d-1$ elements.

Then by Nakayama, $\mathfrak{n}$ can be generated by $d-1$ elements, so S is regular, an thus, by induction, an integral domain. Thus $(x) \subseteq R$ is prime, but not a minimal prime, so it contains some minimal prime Q of R.

If $y \in Q$ then $y = ax$, some $a \in R$. $x \notin Q$ so $a \in Q$. Thus, $Q = xQ = (x)Q$, so by Nakayama $Q = 0$. Thus R is an integral domain. □

A regular system of parameters also satisfies the follow nice property:

Def: A sequence of elements $x_1, \dots, x_d$ in a ring R is an <u>R-sequence</u> or <u>regular sequence</u> if $(x_1, \dots, x_d)$ is a proper ideal and for each i, x_{i+1} is a nonzerodivisor in $R/(x_1, \dots, x_i)$.

In general, a regular sequence may depend on the order of elements.

Ex: In $\mathbb{C}[x,y,z]$, $x, y(1-x), z(1-x)$ is a regular sequence, since $\mathbb{C}[x,y,z]/(x) \cong \mathbb{C}[y,z]$ and $\mathbb{C}[x,y,z]/(x, y(1-x)) \cong \mathbb{C}[z]$.

However, $y(1-x), z(1-x), x$ is not a regular sequence:

$z(1-x)$ is a zerodivisor on $\mathbb{C}[x,y,z]/(y(1-x))$.

However, in a local ring, order doesn't matter.

Cor: If $x_1, \ldots, x_d$ is a regular system of parameters in a regular local ring, then $x_1, \ldots, x_d$ is a regular sequence.

Pf: For each i, the ring $R/(x_1, \ldots, x_i)$ is local of dimension $\geq d-i$. The maximal ideal is generated by $x_{i+1}, \ldots, x_d$, so the dimension $= d-i$, and thus it is regular, and thus an integral domain.

The image of x_{i+1} in $R/(x_1, \ldots, x_i)$ is nonzero (by minimality of generators), so it's a nonzerodivisor. $\square$

Discrete valuation rings

Regular local rings of dimension d correspond to smooth points on schemes of dimension d. In the case of curves $(d=1)$, these rings are called <u>discrete valuation rings</u> (DVRs).

If R is a DVR, then $m=(x)$, for some $x \in R$. x is called a <u>uniformizing parameter</u> for R.

Prop: Let R be a DVR, π a uniformizing parameter, K the field of fractions. Then every element $t \in K$ can be written

$t = u\pi^n$ where $u \in R$ is a unit, and $n \in \mathbb{Z}$. (The map $v : K \to \mathbb{Z}$ defined $t \mapsto n$ is the corresponding valuation.) In particular, every ideal of R is of the form (π^n) and R is a PID.

Pf: By the Krull intersection Theorem, $\bigcap_{i=1}^{\infty} (\pi^i) = 0$, so for $s \in R$, we can choose the largest $m \in \mathbb{Z}$ s.t. $s = v\pi^m$. Thus, $v \notin (\pi)$, so u is a unit.

If $t \in K$, write $t = s_1/s_2 = \dfrac{u_1\pi^{n_1}}{u_2\pi^{n_2}} = \underbrace{u_1/u_2}_{\text{unit in } R}\, \pi^{n_1-n_2}$.

If $t = u\pi^n = v\pi^m$, then $u/v = \pi^{m-n}$ is a unit in R, so $m-n=0$, and $u/v = 1$. Thus, the representation is unique. $\square$